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COMMENT

A connection between the percolation transition and the onset of chaos in the Kauffman model

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Abstract. It is demonstrated numerically that the percolation transition of the unstable sites and the onset of chaos in the Kauffman model happens for the same value of the bias on the randomly chosen rules in the two-dimensional triangular lattice and in the three- and four-dimensional hypercubic lattices. The percolation thresholds for the stable sites are different for the three- and presumably also four-dimensional lattices. On the triangular lattice the differences between the thresholds are too small to be resolved. The fractal dimension of the damage is calculated and also the spreading time on the four-dimensional hypercubic lattice.

Of all the cellular automata [1] one may invent, the Kauffman model [2] is the most disordered one. This automaton was introduced in the biological literature in 1969 to model cell differentiation. Other related questions concerned the stability against mutations. Through this last topic the Kauffman model has become a tool for understanding the general topic of propagation of damage in complex systems, i.e. under what conditions does minor damage spread through the entire system, and when does it stay localised?

In this comment I study numerically the connection between this propagation of damage and percolation theory [3]. A close connection exists between these two concepts, as has been previously demonstrated by Weisbuch and Stauffer [4] and Stauffer [5] for the Kauffman model on a two-dimensional square lattice with nearest-neighbour interactions. I present results in this comment which show that for other lattice types and dimensions some of the results of [4, 5] survive while others do not. I also present here, to my knowledge, the first numerical study of the four-dimensional Kauffman model.

In the Kauffman model each node i belonging to a network with some topology has a Boolean variable σ_i associated with it. The values of these Boolean variables are updated at discrete timesteps by a set of rules

$$\sigma_i(t+1)=f_i(\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{iK}(t)) \qquad (1)$$

where t is the time. Here the updating of the node i is determined by the values of the Boolean variables of K nodes $i_1, i_2, \ldots, i_K$. The set of rules is chosen at random at time $t=0$, and not changed afterwards, thus defining a quenched disorder. A control parameter p governing the disorder may be introduced by biasing the choice of rules in the following way [2]: a fraction p of all the possible configurations of the Boolean variables associated with the input nodes $i_1, \ldots, i_K$ at the time t will result in the

Boolean variable associated with node i taking the value 1 (out of the two possible choices 0 and 1) at time $t+1$. Thus, for $p=0$ or 1, there is no disorder; $\sigma_i(t)=0$ or 1 for all nodes when $t>0$. Maximum disorder is found for $p=\frac{1}{2}$.

Derrida and Stauffer [6] and Weisbuch and Stauffer [4] reported that there is a critical value of the control parameter $p=p_c$ such that for $p>p_c$, called the chaotic phase, damage will spread in the network, while for $p<p_c$, called the frozen phase, any damage will disappear or be confined to a local area of the network. The operational definition of damage here is the Hamming distance between two systems. Initially these systems are chosen identical (i.e. the initial configuration of the Boolean variables and the set of rules chosen are identical) except for one node where the value of the Boolean variable differs. The Hamming distance, M, between the two systems at a given time is simply the number of nodes whose Boolean values differ between the two systems.

At the onset of chaos, $p=p_c$, the Hamming distance M scales with the linear size of the network, L, with a fractal dimension [7] d_f,

$$M \sim L^{d_f}. \tag{2}$$

Furthermore, the time T for the damage to touch the boundaries of the network scales with an exponent [7] d_t,

$$T \sim L^{d_t}. \tag{3}$$

The values of these exponents in two [7, 8], three [9] and four dimensions are given in table 1, together with the values of p_c.

Table 1. The critical probability, p_c, the fractal dimension of the damage and the critical exponent of the damage spreading time for various types of lattices. The superscripts refer to the reference list. This work deals with the four-dimensional data.

Lattice	p_c	d_f	d_t
2D square	0.29[7]	1.5[7]	1.7[7]
2D triangular	0.16[8]	1.5[8]	1.5[8]
3D cubic	0.12[9]	1.8[9]	2.2[9]
4D hypercubic	0.08	1.8	2.1

The values quoted in table 1 for the four-dimensional hypercubic lattice were found by essentially the same method as used by de Arcangelis [9], except that I did not use the multispin coding technique described there since it was necessary to store 256 rules per site and thus it was more memory efficient to store these rules f_i bitwise instead of the values of Boolean variables σ_i. These calculations were done mostly on a Cyber 76 computer, and the updating time per site was 21.5 μs on this machine. First I determined the critical p_c by measuring the fractional damage, $\psi(p, t)$, i.e. Hamming distance, M, divided by the total number of nodes, for various p and for t large enough so that the asymptotic behaviour is reached. The lattice size was $L=7$, and I used helical boundary conditions in three directions. I also ran the program with smaller lattices than $L=7$, and from these data it seems that the results are rather insensitive with respect to the size of the lattice. The data I obtained for $L=7$ are shown in figure 1. They indicate that $p_c=0.080\pm0.005$. A lower bound for p_c has been given by Derrida

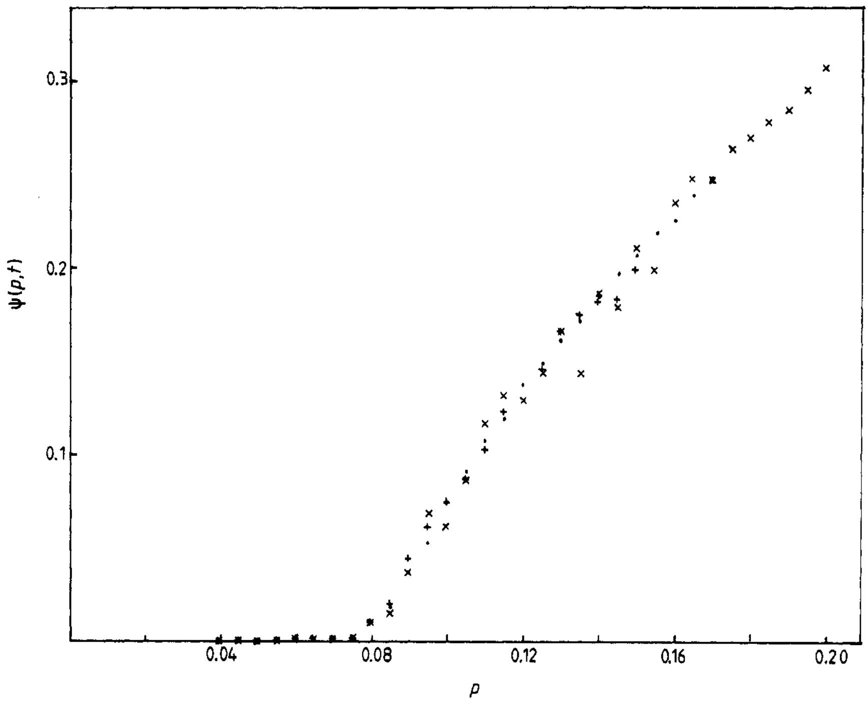

Figure 1. The fractional damage $\psi(p, t)$ as a function of the bias p. The lattice size is $L = 7$ for all the data points. The data points marked · were based on 500 realisations with $t = 100$, those marked + were based on 62 realisations and $t = 500$, and those marked × were based on 10 realisations and $t = 2000$.

and Pomeau [10]: $2p_c(1-p_c) = 1/K$ where K is the number of input sites in the Kauffman rules, equation (1). The formula gives, for $K = 8$, a p_c equal to 0.07. Stauffer [7] has given an upper bound on p_c by the expression $2p_c(1-p_c) = \pi_c$ where π_c is the bond percolation threshold of the lattice. On the four-dimensional hypercubic lattice $\pi_c = 0.16$, which gives $p_c = 0.09$ as an upper bound.

I measured the exponents d_f and d_t by measuring the Hamming distance M and the time T when the damage first reaches the outer boundary of the lattice for lattice sizes varying from $L = 4$ to $L = 9$ when $p = p_c$. These quantities were averaged over 18 600 realisations. M and T are plotted against L in figure 2. A least-squares fit to (2) and (3) gave $d_f \approx 1.8$ and $d_t \approx 2.1$. I dropped the $L = 4$ data for the mass of the damage (see figure 2). It appears that this value is outside what is to be expected as a result of statistical fluctuations, and may be caused by correction-to-scaling terms appearing for this small lattice size. It is curious that these two exponents are essentially the same as those found in three dimensions [9]. This may indicate that the $d = 3$ results have been obtained close to the upper critical dimension for the model.

We now turn to the connection between the percolation critical point and the onset of chaos in the Kauffman model. This topic was investigated by Weisbuch and Stauffer [4] and Stauffer [5] for the two-dimensional square lattice. These authors found that both the stable sites, i.e. those sites whose associated Boolean variable never changes (during the latter half of a run), and the unstable sites, i.e. those sites whose associated Boolean variable does change (during the latter half of a run) have their percolation thresholds [3] coinciding with the onset of chaos. Stauffer also found that the bias p

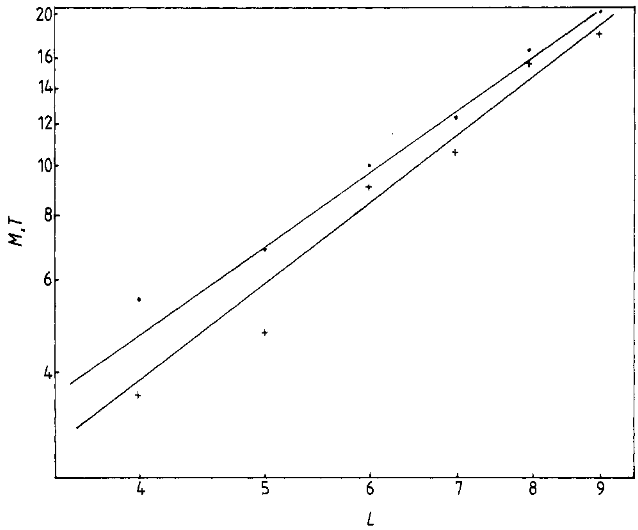

Figure 2. The Hamming distance M (·) and the propagation time T (×) when the damage touches the surface of the network as a function of L for the four-dimensional hypercubic lattice. The runs were done at $p=0.075$.

for which the densities of both stable and unstable sites are equal, coincides with p_c, the onset of chaos.

I have done similar measurements for the two-dimensional triangular lattice, the three-dimensional cubic lattice and the four-dimensional hypercubic lattice. The structure of the percolation clusters was identified by the Hoshen–Kopelman algorithm [11]. More specifically, I measured the percentage of stable and unstable sites belonging to the largest cluster of each type of sites, $P_{s\infty}$ and $P_{u\infty}$. I also measured the average size of the 'finite' clusters (i.e. not counting the largest cluster) for both the stable and unstable sites. The data are graphically represented for the four-dimensional hypercubic lattice in figure 3. I define the percolation thresholds, here called respectively p_s and p_u, for the stable or unstable sites as the p for which $P_{s\infty}$ or $P_{u\infty}$ have their maximum slopes given in table 2. I also record the p, here called p_{su}, for which the density of stable sites equals the density of unstable sites. Furthermore, the p for which the average size of the finite cluster of the stable and unstable sites is maximum I call p_{sm} and p_{um}. All the above defined biases were very insensitive to the lattice size L. However, they were all, with the exception of p_u and p_{um}, sensitive to the number of timesteps I let the system run. A 'finite time' scaling ansatz of the form

$$p(t)=p_\infty+a/\log t \tag{4}$$

seems to work well for the various thresholds [5]. The asymptotic threshold biases found with (4) are given in table 3. They indicate that the percolation thresholds for the unstable sites in the cubic and four-dimensional hypercubic lattices, p_u, seem to be equal to the onset-of-chaos threshold, p_c. In the two-dimensional triangular lattice the various thresholds are too close to be distinguished. The percolation thresholds for the stable sites seem only to be equal to p_c in the two-dimensional lattices [5]. In higher dimensions they are different, even though the four-dimensional data are less clear. However, it may be that a larger computer effort than the present one will show that also some of the two-dimensional lattices have distinct thresholds. The bias for

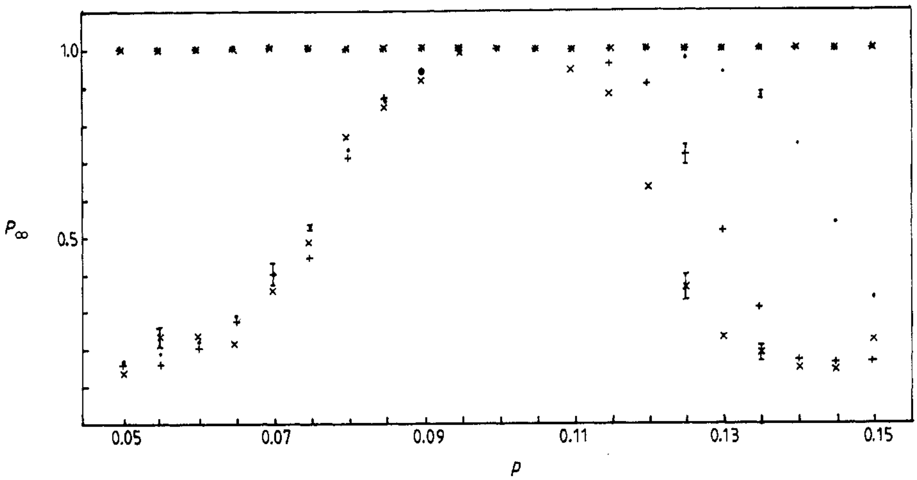

Figure 3. $P_{s\infty}$, and $P_{u\infty}$ for the four-dimensional hypercubic lattice as a function of p for $t = 100$ (·), $t = 500$ (+) and $t = 2000$ (×). The number of realisations used to generate these data is the same as quoted in figure 1. The $P_{s\infty}$ data start in the upper left corner and the $P_{u\infty}$ start in the lower left corner.

Table 2. The effective thresholds for the thresholds defined in the text as a function of the running time.

Lattice	Time	p_s	p_u	p_{su}
	100	0.168	0.161	0.165
2D triangular	400	0.166	0.159	0.164
	1600	0.167	0.158	0.163
	100	0.185	0.122	0.154
3D cubic	500	0.174	0.122	0.150
	2000	0.169	0.122	0.147
	100	0.144	0.074	0.110
4D hypercubic	500	0.130	0.074	0.104
	2000	0.120	0.074	0.102

Table 3. The asymptotic values of the thresholds defined in the text as found by assuming a time dependence as in (4).

Lattice	p_s	p_u	p_{su}
2D triangular	0.165 (5)	0.155 (5)	0.160 (5)
4D hypercubic	0.085 (5)	0.074 (5)	0.090 (5)

which the density of stable and unstable sites are equal, p_{su}, seems to be equal to p_s. The percolation thresholds, p_{sm} and p_{um}, based on the maximum of the mean cluster sizes are consistent with those based on $P_{s\infty}$ and $P_{u\infty}$.

This points towards a close connection between percolation theory and the onset of chaos in the Kauffman model, though not as close as one might first have guessed from the two-dimensional data only [6]: in general the onset of chaos agrees with the percolation threshold of the unstable sites, and disagrees with the percolation threshold of the stable sites.

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